College and Career Readiness Standards for Mathematics

Selecting and Sequencing Student Solutions

Facilitating productive mathematics discussions in the classroom.

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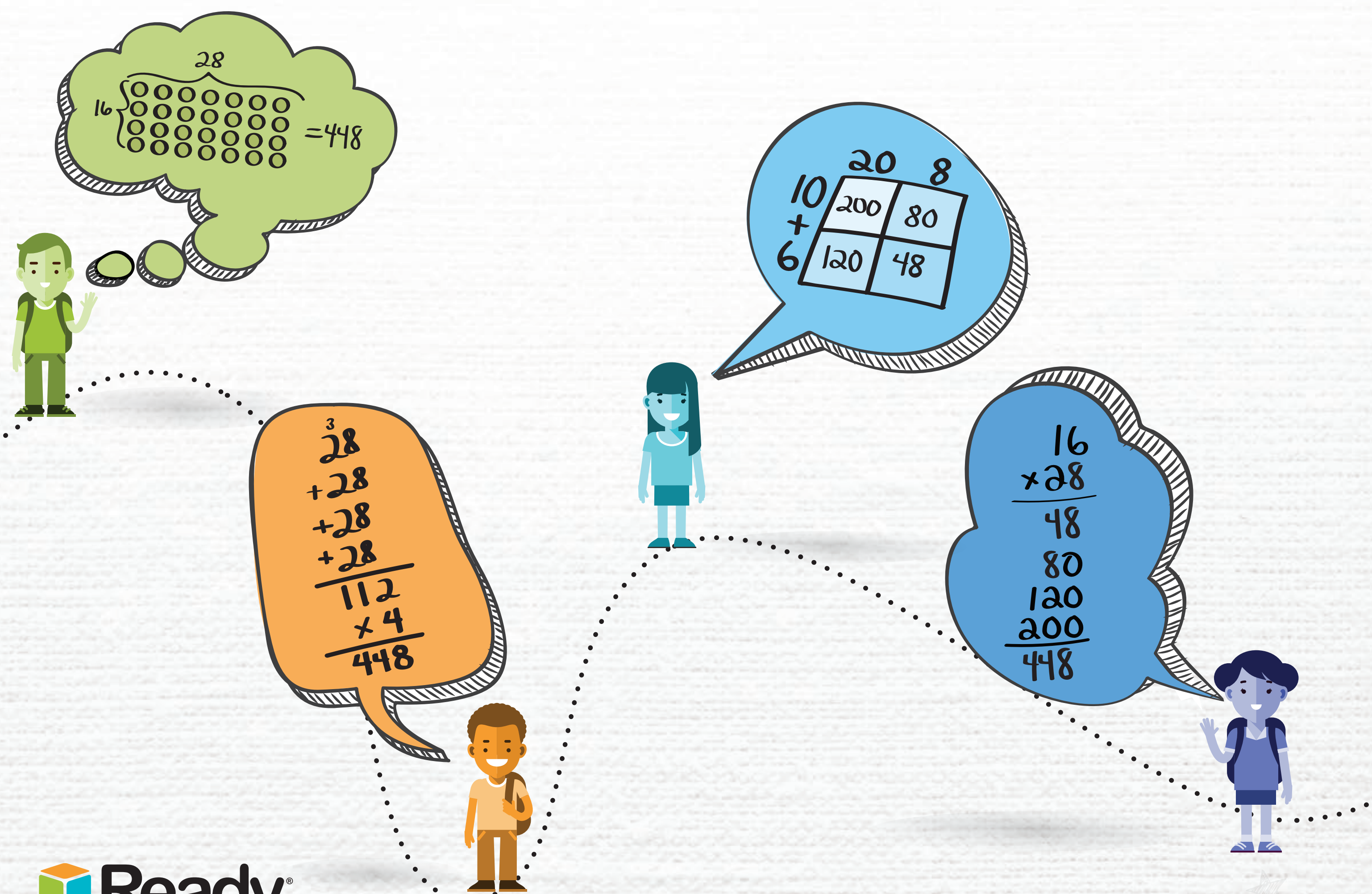

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About the Author

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Gladis is the **Dean of the Neag School of Education at the University of Connecticut** and a respected scholar in mathematics education. She has published books, chapters, and journal articles on teacher education, effective teaching of at-risk students, and the use of technology in teaching and learning. She was a member of the Board of Directors for both the National Council of Teachers of Mathematics (NCTM) and the Association of Mathematics Teacher Educators. Her work is driven by the belief that all students deserve equitable access and opportunities to learn mathematics so they can benefit from the options education provides.

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Introduction

The important role mathematical discourse plays in helping students make sense of mathematics is well established. The National Council of Teachers of Mathematics (NCTM, 2014) cited earlier standards documents (NCTM, 1991; NCTM, 2000) when it asserted, *"Effective mathematics teaching engages students in discourse to advance the mathematical learning of the whole class. Mathematical discourse includes the purposeful exchange of ideas through classroom discussion, as well as through other forms of verbal, visual, and written communication. The discourse in the mathematics classroom gives students opportunities to share ideas and clarify understanding, construct convincing arguments regarding why and how things work, develop a language for expressing mathematical ideas, and learn to see things from other perspectives."* (p. 29)

The focus on mathematical discourse reflects a shift in thinking regarding *what* mathematics students are expected to know and *how* they are expected to know it. To be productive twenty-first-century citizens, students must understand mathematics in ways that permit them to engage fully with the subject matter, understand the concepts that underpin mathematical ideas, and apply mathematics to a variety of mathematical and real-world problem solving contexts. It is by engaging students in mathematical discourse that teachers are able to *access* and *assess* what students know so they can engage in purposeful planning to facilitate student learning.

Research shows that there are best practices that can be used to create a classroom environment that fosters rich classroom discussions. Stein and Smith (2011) detail five specific practices teachers can use to plan and lead productive mathematics discussions. (See sidebar at right.) The five practices are viewed as critical components of a well-designed plan to make purposeful and thoughtful decisions that, ultimately, strengthen students' understanding of robust mathematics.

The Five Practices That Promote Classroom Discussion

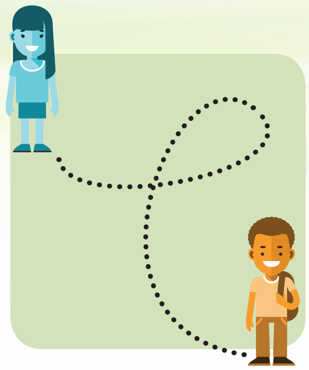
1. **Anticipating.** *"Actively envision how students might approach the mathematics task they will work on"* (p. 8). Thinking about student approaches, possible errors, and misconceptions allows teachers to better plan their questioning strategies.

2. **Monitoring.** *"[Pay] close attention to students' mathematical thinking and solution strategies as students work the task"* (p. 9). Did any students use strategies you did not think about? Did any use a visual model or more sophisticated strategy that you want to highlight with the class? Did any make errors that highlight a common misconception you want to highlight with the class?

3. **Selecting.** *"Select particular students to share their work with the rest of the class to get specific mathematics into the open for examination"* (p. 10). The selected students can be alerted in advance to give them time to gather and organize their thoughts.

4. **Sequencing.** *"Make decisions regarding how to sequence the student presentations"* (p. 10). The goal is to maximize the connections between and among ideas. For example, a teacher may first call on a student or group with incorrect thinking or an incorrect answer to highlight a common misconception before the class discusses the correct answer.

5. **Connecting.** *"Help students draw connections between their solution and other students' solutions as well as the key mathematical ideas in the lesson"* (p. 11). This synthesis will help reinforce and extend learning.



Planning for Productive Mathematics Discussions

Prior to engaging in the five practices, it is important to thoughtfully identify a task or problem that is conducive to mathematics discussion. The task can't be selected casually—it should provide students at various levels of understanding an opportunity to begin working on the problem using a variety of approaches and representations. Teachers should ask themselves the following questions when choosing a task:

- What **instructional goals** do I hope to achieve by using this problem? What **understanding** do I wish students to develop?
- Is the intent of the lesson to **develop a new concept or assess student understanding of previously learned concepts**?
- What **behaviors, norms, or routines** am I hoping to introduce or reinforce?
- What are the **communication goals** for the lesson (e.g., use of mathematics language, written expression of mathematics ideas, etc.)?

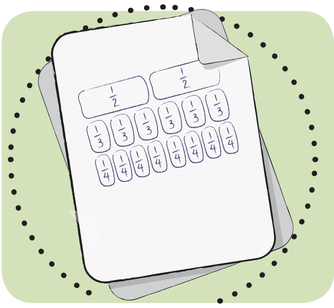
By answering these questions, teachers can think about the goals of the lesson and what they hope to achieve during the lesson, which will lead to better lesson planning and ultimately, improved instruction. The answers to the questions, particularly the first question, may depend on many factors including the grade level at which the task is used. For example, in fourth grade, students may see a problem like this:

> *Folding chairs are set up in a school auditorium for a play. There are 16 rows of chairs, each with 28 chairs. How many folding chairs are there?*

With this task, the teacher's goal is to determine if students can apply strategies from previous lessons—in this case, multiplying a four-digit number by a one-digit number using place value, properties of operations, rectangular arrays, area models, and/or equations—to a similar, but new situation (multiplying two two-digit numbers). The teacher may ask students to solve the problem using two or more representations or strategies that they have explored previously. It is important to identify the behaviors, norms, and routines to introduce or reinforce in the classroom, because without them, it will be difficult for students to extend their focus beyond just the mathematics. If students are not accustomed to engaging in mathematics discussions, for example, a teacher might use a low-threat task to build students' confidence in sharing their solution strategies with a partner or small group before sharing with the whole class. If the class has engaged in mathematics discussions as a normalized classroom routine, the teacher might use the task to encourage students to *generate viable arguments* (solution strategies) and engage students in the process of *critiquing the reasoning of others* (Standard for Mathematical Practice 3).

It is important to ***identify the behaviors, norms, and routines to introduce or reinforce*** *in the classroom, because without them, it will be difficult for students to extend their focus beyond just the mathematics.*

After the instructional goals are set, teachers can start to plan for and anticipate the types of strategies students might use to solve problems and the likely responses they will give to questions. Anticipating students' responses helps teachers plan productive ways to engage students depending on the generated responses and strategies that come up as part of class discussion. As teachers monitor students as they work, they may notice a number of students using the same strategy or may find a student who is using a unique strategy—one that is unlike what the teacher anticipated or other students used. Once the teacher identifies these solution patterns, she must next consider which ones will be shared and in which order.



Selecting Student Work for Classroom Discussion

According to Stein and Smith (2011), "selecting is the process of determining which ideas (what) and students (who) the teacher will focus on during the discussion" (p. 43). Deciding which students will share their strategies is an important decision. In cases in which multiple students use the same strategy, the teacher must decide which student to call on to share his or her approach. Some teachers might opt to call on a student who is confident and can express himself or herself well. In other cases, a teacher might provide additional support and reassurances to prepare less confident students for this role (e.g., "I want you to share your strategy because I think it might help others in the class. Let's go over what you will say so that you are comfortable speaking to the group.").

Let's take a look at sample student work for the task discussed in the chair example on page 5, which is organized and discussed in three different groups:

Group 1 Several students used similar solution strategies, but represented them in slightly different ways. Figure 1 (p. 7) illustrates how students **visually modeled** the problem using discrete or area representations.

Group 2 Others used their understanding of **place value** to find a solution (see Figure 2, p. 7).

Group 3 Some students used a variety of computational approaches, including **the use of multiples or repeated addition** (see Figure 3, p. 8).

*Deciding **which students will share their strategies** is an important decision.*

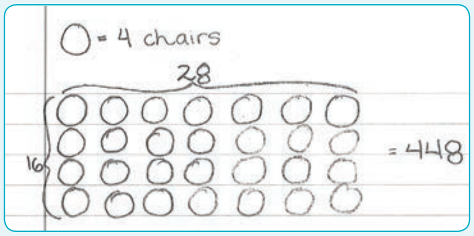
Note that in some cases the students identified the correct answer, but it is not clear how they determined the result. Before moving on to the next section, consider which student work you would select for classroom discussion if you were to see this collection of solutions in your classroom. Here are some things to consider when making your selections:

- What might be the usefulness of selecting work that is incomplete or incorrect?
- For students who used the same or a similar strategy, consider which example provides the most transparency about the students' thinking or reasoning.
- If you have time to discuss only two or three examples, which ones will you select to ensure a productive discussion that furthers students' understanding and connects to the goals of the lesson?
- Given the various approaches used by students, how would you sequence the class discussion to support productive conversations? What questions would you ask and why?
- What basis would you use for determining which students you would call on?

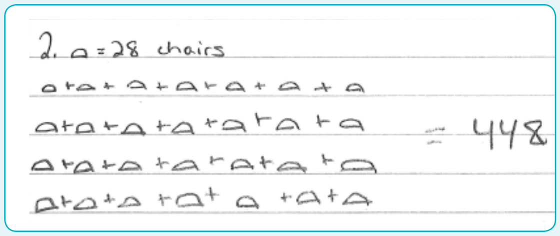
Figure 1: Visual Illustrations

Several students visually illustrated the problem using discrete or area model representations.

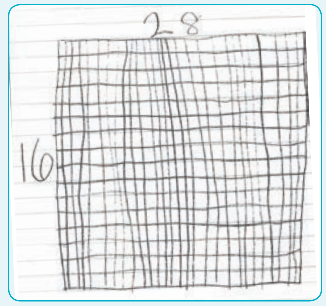
Robin



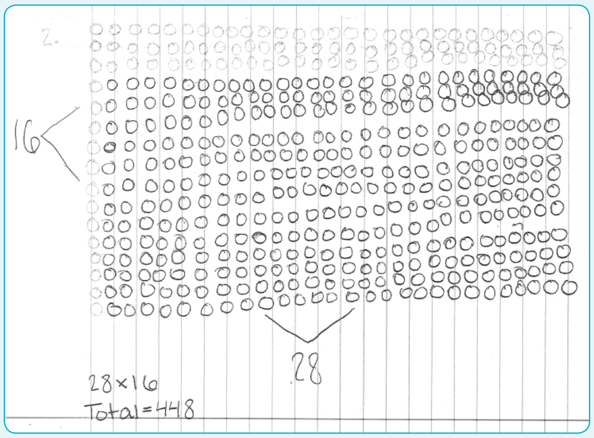
Francis



Joaquin



Taylor



Raul

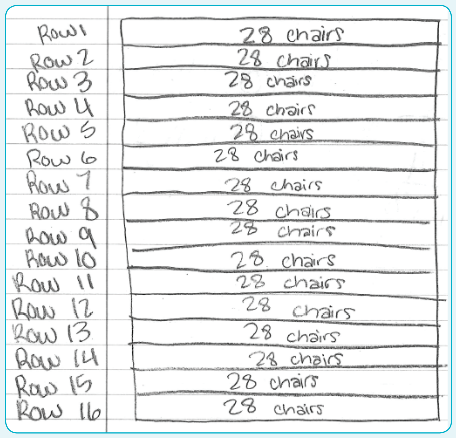
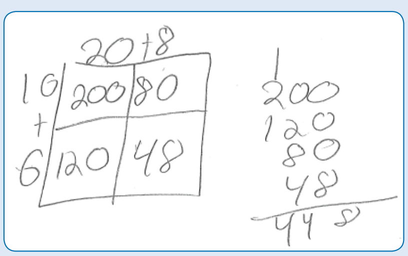


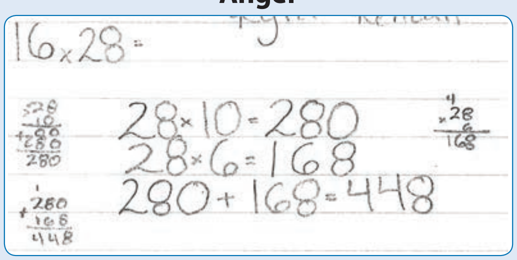
Figure 2: Place Value Representation

Some students used their understanding of place value to find a solution.

Juan



Angel



Raashid

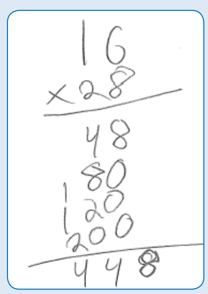
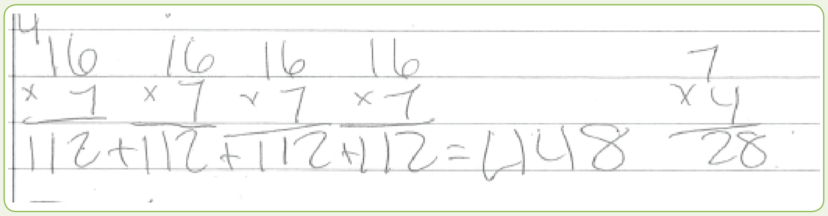


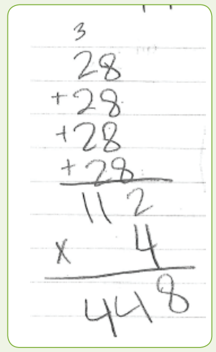
Figure 3: Computational Approaches

Some students used a variety of computational approaches, including the use of multiples or repeated addition.

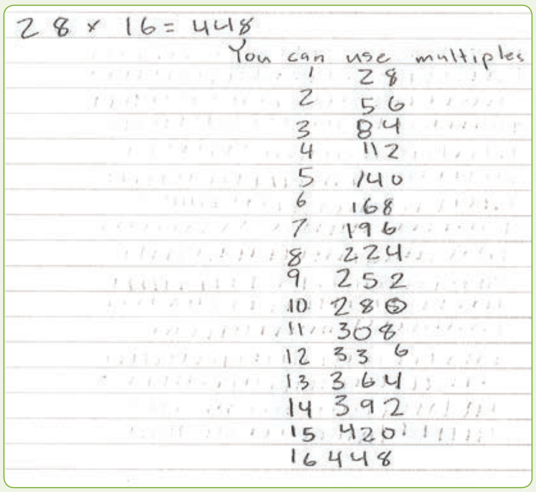
Jordan



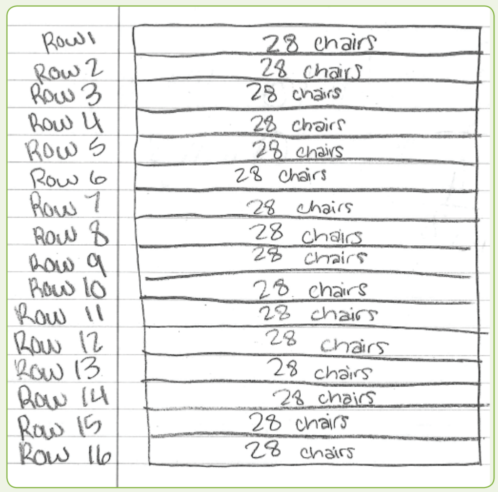
Aisha

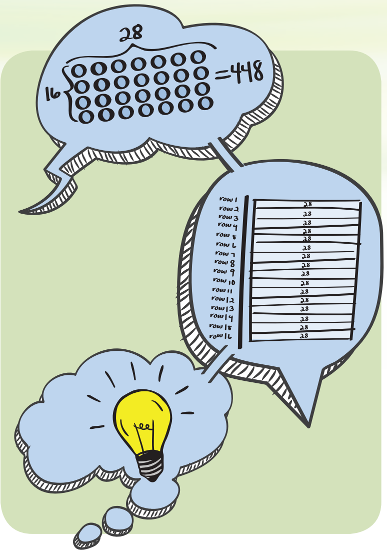


Lei



Raul





Sequencing Student Work to Support Meaningful Classroom Discussion

There are a number of approaches teachers might take when determining how to sequence students' presentations. It might be appropriate to move from approaches that are less mathematically sophisticated to those that are more sophisticated (in terms of the mathematical understanding required) to encourage students to consider more efficient strategies. The same is true when examining written solutions that may include visual representations, words, and symbols. The goal is to help students identify advantages and hindrances that may be apparent in their own verbal or written responses and consider alternative approaches for showing their work. In addition, asking a student to share a unique strategy may help other students to think more flexibly and learn different ways to solve a problem or represent their own results.

Although teachers often ask students who have produced correct results to share their strategies, there is much to be learned from students who make mistakes. That is, errors can be used as a springboard to initiate discussions with questions such as, "Do you agree or disagree with Andi's strategy? Why or Why not?" By identifying, sharing, and discussing the causes of errors, students learn to avoid potential pitfalls and misconceptions that may interfere with their reasoning and understanding. In fact, as students work through errors they see the value of reflecting on the processes they used and come to see errors as a natural part of the learning process. Such engagement with errors gives students permission to take more risks in the mathematics classroom where the focus is on learning and not just on producing the correct answer.

Now we will look at a possible sequence for encouraging mathematical discussions and solution comparisons using the student work shown earlier.

By identifying, sharing, and discussing the causes of errors, students learn to avoid potential pitfalls *and misconceptions that may interfere with their reasoning and understanding.*

Strategies using visual representations of the situation

Figure 1 (p. 7) illustrates a group of similar strategies; so a teacher could begin by first comparing the visual representations used by Taylor and Joaquin. Although they both represented every chair in each row, Taylor used a discrete model and Joaquin used an area model. Although either approach could be used to determine the answer, it would be beneficial to ask a question such as,

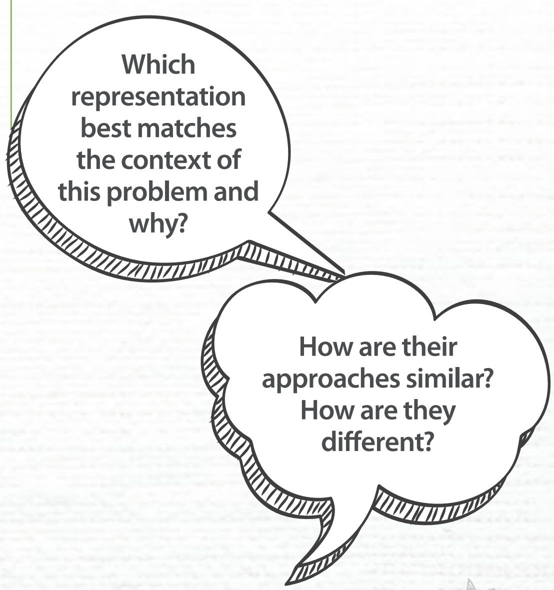
> *Which representation best matches the context of this problem and why?*

Next, a teacher could ask students to compare and contrast Joaquin's and Raul's representations.

> *How are their approaches similar? How are they different?*

Raul uses a process similar to Joaquin's, but instead of drawing a representation for each chair, he simply writes "28 chairs." Teachers can further facilitate classroom discussion by asking:

> Which approach is more efficient—or requires the least amount of visual effort? Why?



For all three of these solution strategies, students should be asked to explain how they obtained their solutions. For example, it appears that these students used a counting strategy (i.e., counting one by one) or a repeated addition strategy (i.e., adding 28 repeatedly, 16 times) to determine the total, but until students are asked to express their thinking, it is impossible to know.

It is essential for students to explain their work and why they chose the method they used and for teachers to ask clarifying questions. Instructional routines that promote student discourse and ask students to compare strategies, such as *Ready's* Think-Share-Compare routine, provide teachers with a classroom structure that will encourage and support these rich classroom conversations. Teachers may examine students' work and assume that they can interpret students' approaches, but this assumption may be erroneous. Depending on a student's ability to express him- or herself mathematically, what is illustrated on paper may represent only part of the picture. That is, there may be aspects of the student's thinking that were not captured by the illustrations provided, or that were not well explained because the student is still developing in this area. It is only when teachers ask a student to explain his or her reasoning to another student or to the whole class that a student exposes what he or she was thinking, which provides teachers opportunities to support student learning and improve students' ability to express their mathematical ideas.

Consider Robin's solution in Figure 1 (p. 7), which appears to be similar to Taylor's, but each circle in her solution is intended to represent 4 chairs. In this case, the teacher might gain insights into Robin's thought processes by asking questions such as,

- ***How are you representing the 28 chairs in each row?***
- ***How are you representing the 16 rows?***

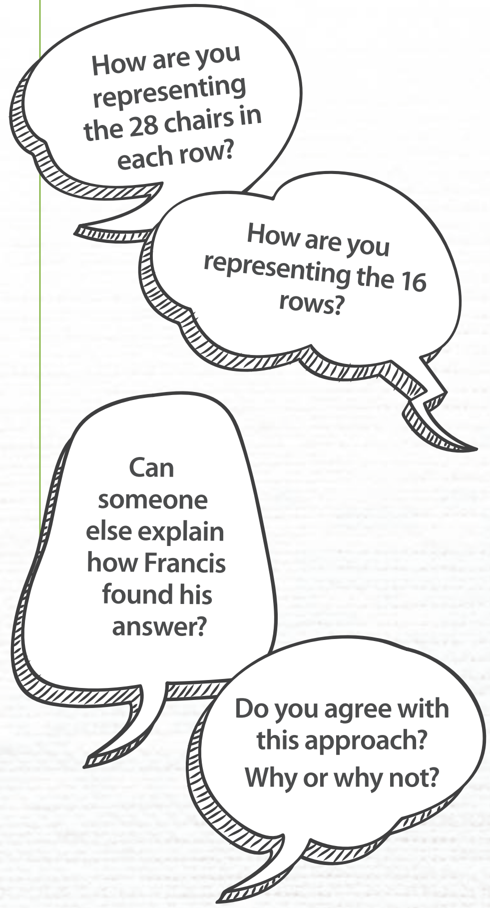
This should spur Robin or others to recognize that there is an error in her representation and encourage her to revisit her reasoning.

Let's look at another example of comparing different visual representations using Francis's solution. After students see Francis's solution and hear his explanation of his solution strategy, a teacher can ask the other students if they understood his strategy with questions like:

- ***Can someone else explain how Francis found his answer?***
- ***Do you agree with this approach? Why or why not?***

Depending on the responses provided by students, the teacher can guide students to detect the error in Francis's representation even though he arrived at the correct answer. Asking Francis to explain what he intended to convey reinforces the importance of using visual representations that accurately depict the student's line of reasoning.

Classroom routines that ***promote student discourse and ask students to compare strategies****... provide teachers with a classroom structure that will encourage and support these rich classroom conversations.*



Strategies using place value

Following the discussion of the visual representations, the class could engage in a discussion of the approaches that utilize students' understanding of place value, as shown in Figure 2 (p. 7).

Let's look more closely at Juan's, Angel's, and Raashid's solutions. Juan decomposes 28 and 16 using place value and then connects multiplying the binomials (20 + 8) and (10 + 6) with the partial products algorithm. He illustrates this approach by using an area model that shows the product of each set of numbers in the binomials and then shows how he combined the partial products to determine the final solution of 448. Angel and Raashid use similar strategies. Angel decomposes 16 into 10 + 6 and then multiplies the binomial (10 + 6) by 28 to determine the result. Raashid uses a more traditional multiplication procedure, but illustrates the value of each digit in the process by finding four partial products and adding these to find the result (i.e., 8×6, 8×10, 20×6, 20×10). In this case, the teacher might ask each of these students to share his approach and then have the class compare the solution strategies presented. Questions the teachers might ask include:

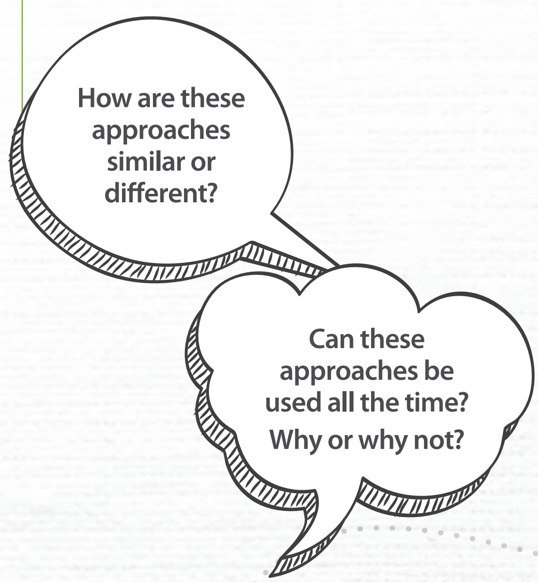
- *How are these approaches similar or different?*
- *Can these approaches be used all the time? Why or why not?*

Strategies that break the problem into smaller chunks

Several students broke the problem into smaller chunks and used repeated addition to find their solution. To make sense of their approaches, teachers should encourage each student to share his or her solution strategies.

For example, in Figure 3 (p. 8), Aisha used addition to determine the total number of chairs in four rows (112) and then multiplied that value by 4, which indicates that she determined that there are 4 groups of 4 rows. A review of Jordan's work suggests that she divided the number of chairs in a row into four groups of 7. Then, she determined the total number of chairs if there were 7 chairs in each of 16 rows. Because there were 28 chairs in each of the rows, not 7, she multiplied her result by 4 to determine the final answer. As in the previous cases, the teacher can ask Aisha and Jordan to share their solution strategies, and then ask the class to look for similarities and differences in their approaches as part of the classroom discussion.

To make sense of their approaches, ***teachers should encourage each student to share his or her solution strategies.***



Strategies that use computational approaches

The final group of strategies might focus on Lei's and Raul's solutions as shown in Figure 3 (p. 8).

Lei appears to have first attempted to count each chair (see the rows of semi-erased hash marks on her paper), and then recognized that for each row she would need to draw 28 chairs. In her written response, she indicates, "you can use multiples." In this case, it would be important to ask Lei to explain her thought processes. Because there are remnants of a prior attempt in her work, she could be asked to explain why she abandoned her previous approach.

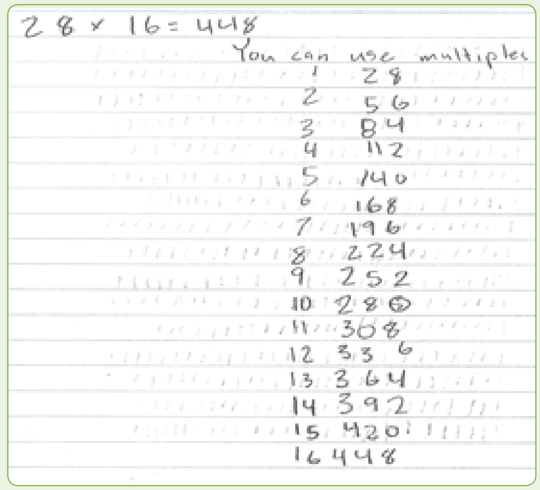
> ***It looks like you started to use one strategy and then switched to another. Why did you decide to switch?***

> ***How did you use multiples to solve this problem?***

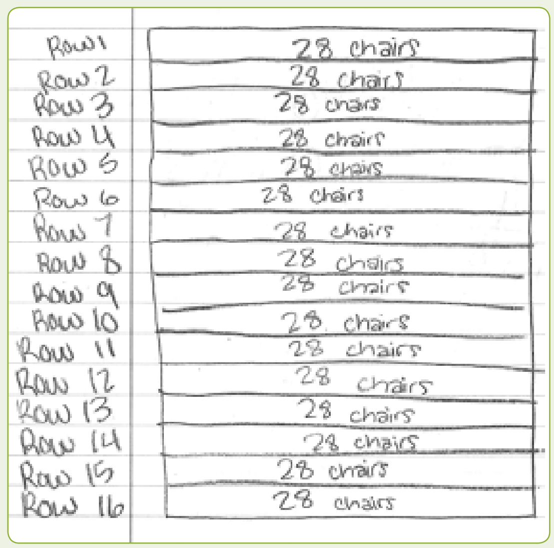
In particular, one would anticipate her response to the first question to indicate that she abandoned the less sophisticated approach when she recognized that there was a pattern she could use. After she explains how she used multiples, her approach can be compared to Raul's. In both cases, the students represent the 16 rows; however, Lei appears to use multiples to document the cumulative total number of chairs as she moves from row to row, whereas Raul found the total number of chairs in each row and then added these to find the total. Although it is possible to infer what students might have been thinking based on their illustration or strategy, a teacher never really knows what a student is thinking unless the student communicates their thinking orally or in writing!

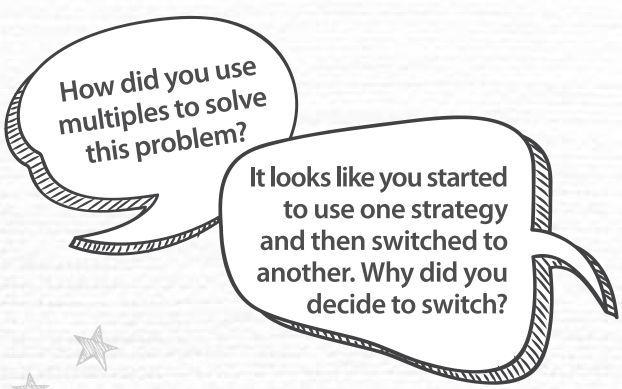
Lei's and Raul's Solutions from Figure 3 (p. 8)

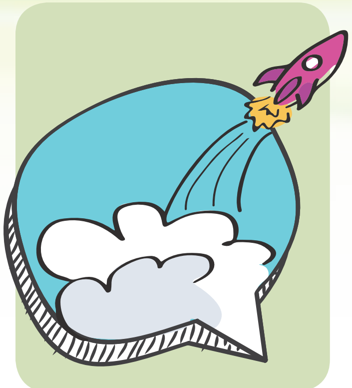
Lei



Raul







Pulling It All Together

Although there are a variety of decision-making possibilities when selecting and sequencing students' work to facilitate meaningful and productive mathematics discussions, the sequence described above focuses on having students consider differences between less to more sophisticated mathematical strategies and visualization approaches. In planning how to sequence student solutions for discussion, it is important for the teacher to help students recognize what information they can or cannot infer from the provided representations and strategies. In the process, students learn how others might interpret the information and are encouraged to consider ways to be more clear and precise when explaining their mathematical reasoning and thought processes orally and in writing.

Pacing classroom discussion

Depending on the students in the class and the amount of time a teacher has, it might be possible to discuss all of these issues in one class period or over several days. The difference in time may be attributed to other decisions a teacher might make. For example, after discussing the different approaches illustrated in Figure 2 (p. 7), a teacher might provide students with additional problems and ask them to represent them using at least two of the different approaches discussed. This will provide students time to try some of the ideas expressed during the discussion, which is especially important if they are directly related to the standard(s) the teacher is trying to address in the lesson.

When students share, discuss, and compare strategies, ***teachers have greater insight into students' thinking.***

As an alternative, a teacher might want to take the time to discuss almost all of the possible approaches to help students see the array of possibilities when solving a problem prior to assigning additional problems to solve. This also helps teachers assess where students are in their thinking and may help further uncover common errors and misconceptions. If there is limited time, teachers may want to have students compare their strategies to those in their instructional materials. Making these comparisons can also help students make connections between the strategies they understand and those that may be new to them. If necessary, the teacher can ask questions to highlight key connections, new strategies, and key concepts.

Asking students to share and compare solution strategies, representations, and oral and written communication approaches provides insights to both students and teachers. Students become active participants in learning mathematics and develop strategies they can use to improve their own mathematical thinking. They learn to ask themselves the questions that they are asking of each other, to explain their thinking, and to critique the reasoning of others. As a result, they reflect on their own thought processes, the strategies shared by others and, consequently, attempt and adopt new approaches. They make sense of problems and persevere in solving them (SMP 1), reason abstractly and quantitatively (SMP 2), construct viable arguments and critique the reasoning of others (SMP 3) and improve the clarity and precision of the ways they convey their solutions (SMP 6). As students connect mathematical ideas, problem solving strategies, and representations, they deepen their own understanding of core mathematics concepts and enhance their ability to express their mathematical understanding orally and in writing (NCTM, 2014).

When students share, discuss, and compare strategies, teachers have greater insight into students' thinking. They are not making assumptions about student understanding based on their interpretation of a solution—they can hear students' explanations and use questions to gain greater insight as to what students know and don't know. Teachers can see how well the class as a whole understands the math concepts being discussed, as well as which students are using more basic approaches to solving problems and which are using more sophisticated or unique strategies. They can identify errors and misconceptions in student thinking, use questions to guide students to recognize and discuss their mistakes, and help students see that mistakes help advance our understanding.

Questions to Consider when Selecting and Sequencing Student Strategies

To help teachers plan for classroom conversations and anticipate possible solutions, use questions like the ones below to help guide you in selecting and sequencing student strategies. Select key questions to consider as you observe student strategies in the classroom.

...when selecting student work:

- What **patterns** am I noticing in students' work?
- What do the identified patterns represent in terms of students' mathematical understanding?
- What are the **unique strategies or approaches** used by students? Will they always work?
- What **mathematics ideas and lesson objectives** can I highlight or reinforce as a result of bringing this example to the class's attention?
- **What errors do I notice?**
- How can I use these errors as springboards for additional learning?

...when selecting which student(s) to call on:

- How do I ensure that **every student has a voice** in my classroom?
- What **supports** do I need to provide to prepare students to experience success in sharing their ideas?
- How do I **structure the learning environment** so that students are willing to **expose their mistakes** and, as a result, learn from them?
- How do I make math conversations a regular part of classroom interactions?

...when deciding how to sequence student explanations:

- What are the **advantages/disadvantages** of sequencing the discussion in this order?
- How do I **build students' understanding** as we transition from one set of solution strategies to the next?
- What **questions do I ask** to encourage students to critique ideas that are presented?
- How do the questions support students in **making connections** between and among the work to be shared?

Conclusion

Even when teachers use routine tasks, there are ample opportunities to engage students in thinking and talking about their strategies and advancing their understanding of mathematics. Teachers need to call attention to students' reasoning and encourage them to provide explicit details about their thought processes verbally and/or in writing. This provides access to and insights about students' mathematical development. In addition, by providing opportunities for students to critique each other's work, they are provided the opportunity to validate their approaches as they see how others may have used the same or a similar approach. They build confidence as they defend their ideas and learn to improve as they see and recognize shortcomings in their own or others' work.

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